

Final Exam: MTH 111, Fall 2017

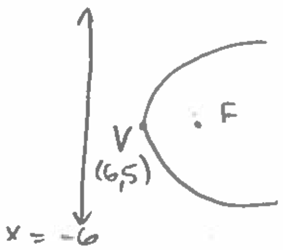
Ayman Badawi

Katia

Points = $\frac{81}{82}$

QUESTION 1. (6 points) Given $x = -6$ is the directrix of of a parabola that has the point $(6, 5)$ as its vertex point.

a) Find the equation of the parabola



$|VL| = |-6-6| = |-12| = 12$

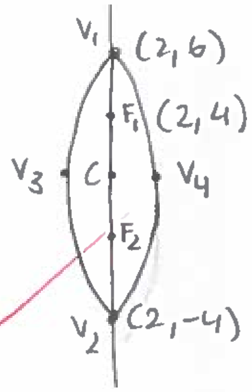
$4(12)(x-6) = (y-5)^2 \Rightarrow 48(x-6) = (y-5)^2$

b) Find the focus of the parabola.

$|VF| = 12 \rightarrow F(18,5)$

QUESTION 2. (8 points) Given $(2,-4), (2,6)$ are the vertices of the major axis of an ellipse (recall major axis is the longer axis) and $(2,4)$ is one of the foci.

(i) Find the vertices of the minor axis (shorter axis). (you may want to draw such ellipse (roughly)).



$|V_1V_2| = K = |6+4| = 10 \rightarrow \frac{K}{2} = 5 = |V_1C|$

$C = (2,1) \rightarrow |F_1C| = |4-1| = 3 \rightarrow b^2 = \left(\frac{K}{2}\right)^2 - |F_1C|^2$

$b^2 = 5^2 - 3^2 = 16 \rightarrow V_3(18,1), V_4(-14,1)$

(ii) Find the ellipse-constant K.

$K = 10$

(iii) Find the second foci of the ellipse.

$F_2(2,-2)$

(iv) Find the equation of the ellipse.

$$\frac{(x-2)^2}{16} + \frac{(y-1)^2}{25} = 1$$

QUESTION 3. (5 points) Given $y = 3x^2 + 12x + 9$ is an equation of a parabola. Write the equation of the parabola in the standard form and find the equation of its directrix.

-1

$y = 3x^2 + 12x + 9 \rightarrow y = 3(x^2 + 4x + 3) \rightarrow y = 3\left[(x+2)^2 - 4 + 3\right]$

$y = 3(x+2)^2 - 1(3) \rightarrow \frac{1}{3}(y+3) = (x+2)^2$

$4d = \frac{1}{3} \rightarrow d = \frac{1}{12}$

$V = (-2,-3) \rightarrow$ directrix $\rightarrow x = -2 - \frac{1}{12} \Rightarrow \boxed{-\frac{25}{12} = x}$

$y = -3 - 1/12$

QUESTION 4. a) (4 points) Given two lines $L_1 : x = 2t, y = -2t + 3, z = -t + 1$ and $L_2 : x = -4w - 12, y = 4w + 15, z = 2w + 7$. Is L_1 parallel to L_2? EXPLAIN clearly.

$L_1 : \left.\begin{array}{l} x = 2t \\ y = -2t+3 \\ z = -t+1 \end{array}\right\} t \in \mathbb{R}$

$L_2 : \left.\begin{array}{l} x = -4w-12 \\ y = 4w+15 \\ z = 2w+7 \end{array}\right\} w \in \mathbb{R}$

$\rightarrow$ $D_1 = \langle 2, -2, -1 \rangle$, $D_2 = \langle -4, 4, 2 \rangle$

$D_2 = C D_1 \rightarrow C = -2 \rightarrow D_1 \parallel D_2$

intersection : $L_1 \rightarrow t = 0 \rightarrow Q(0,3,1)$ — Q Lies on L_2

$L_2 \rightarrow x: 0 = -4w - 12 \rightarrow w = -3$ / $z: 1 = 2w + 7 \rightarrow w = -3$

$y: 3 = 4w + 15 \rightarrow w = -3$ / $\rightarrow$ L_1 not $\parallel L_2$ they overlap

b)(4 points) Let L be the line L_1 as in (a). Given that the point $Q = (2,3,4)$ does not lie on L. Find $|QL|$ (distance between Q and L).

$I = (0,3,1)$, $Q(2,3,4) \rightarrow \overrightarrow{IQ} = \langle 2, 0, 3 \rangle$

$|QL| = \dfrac{|\overrightarrow{IQ} \times D_1|}{|D_1|}$, $\overrightarrow{IQ} \times D = \begin{vmatrix} i & j & k \\ 2 & 0 & 3 \\ 2 & -2 & -1 \end{vmatrix} = \langle 6, 8, -4 \rangle$

$\hookrightarrow |QL| = \dfrac{\sqrt{6^2 + 8^2 + (-4)^2}}{\sqrt{2^2 + (-2)^2 + (-1)^2}} = \boxed{\dfrac{2\sqrt{29}}{3}}$

c) (6 points) Convince me that $q_1 = (1,4,2)$, $q_2 = (2,1,-1)$, and $q_3 = (3,5,2)$ are not co-linear. Then find the area of the triangle with vertices q_1, q_2, q_3.

$\overrightarrow{q_1q_2} = \langle 1, -3, -3 \rangle$

$\overrightarrow{Q_1Q_3} = \langle 2, 1, 0 \rangle$

$\rightarrow \overrightarrow{Q_1Q_2} \times \overrightarrow{Q_1Q_3} = \begin{vmatrix} i & j & k \\ 1 & -3 & -3 \\ 2 & 1 & 0 \end{vmatrix} = \langle 3, -6, 7 \rangle \Rightarrow$ $\overrightarrow{Q_1Q_3}$ not $\parallel \overrightarrow{Q_1Q_2}$ $\Rightarrow$ not colinear

$A_\Delta = \frac{1}{2} |\overrightarrow{Q_1Q_2} \times \overrightarrow{Q_1Q_3}|$

$= \frac{1}{2}\sqrt{3^2 + (-6)^2 + 7^2}$

$= \dfrac{\sqrt{94}}{2}$ units^2

d)(6 points) The two planes $P_1 : 2x + y + 2z = 2$ and $P_2 : -x + y - z = 5$ intersects in a line L. Find a parametric equations of L.

$N_1 = \langle 2, 1, 2 \rangle$, $N_2 = \langle -1, 1, -1 \rangle$

$D = N_1 \times N_2 = \begin{vmatrix} i & j & k \\ 2 & 1 & 2 \\ -1 & 1 & -1 \end{vmatrix} = \langle -3, 0, 3 \rangle$

$\rightarrow$ let $z = 0 \rightarrow 2x + y = 2$, $-1 \times [-x + y = 5]$ $\rightarrow$ $2x + \cancel{y} = 2$, $x - \cancel{y} = -5$

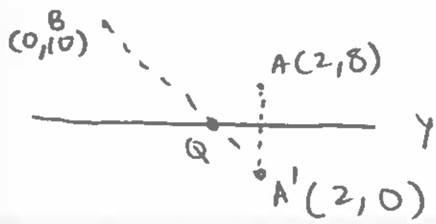
$3x = -3 \rightarrow \boxed{x = -1} \rightarrow 2(-1) + y + 2(0) = 2$

$\boxed{y = 4}$

$Q = (-1, 4, 0)$

$\rightarrow \boxed{L_1 : \left.\begin{array}{l} x = -3t - 1 \\ y = 4 \\ z = 3t \end{array}\right\} t \in \mathbb{R}}$

QUESTION 5. (6 points) Let $A = (2,8)$, $B = (0,10)$. Find a point Q on the line $y = 4$ such that $|BQ| + |QA|$ is minimum.



$|AL| = |8 - 4| = 4$

$\rightarrow m = \dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{10 - 0}{0 - 2} = \dfrac{10}{-2} = -5$

$y = -5x + b \rightarrow 10 = -5(0) + b \rightarrow b = 10$

$y = -5x + 10 \rightarrow 4 = -5x + 10 \rightarrow 4 - 10 = -5x \rightarrow x = \dfrac{+6}{+5}$

$\boxed{Q = \left(\frac{6}{5}, 4\right)}$

QUESTION 6. (9 points)

(i) Given $f'(1) = 2$ and $y = f(x^2 + 2x - 7)$. Then $y'(2) =$

$$y' = \left[f'(x^2+2x-7)\right]\left[2x+2\right] = \left[f'(2^2+2(2)-7)\right]\left[2(2)+2\right] =$$

$$\left[f'(1)\right]\left[6\right] = 6(2) = \boxed{12}$$

(ii) Let $f(x) = -6e^{(x^3+6x-7)}$. Then $f'(x) =$

$$f(x) = -6e^{(x^3+6x-7)} \rightarrow f'(x) = -6(3x^2+6)(e^{x^3+6x-7})$$

$$f(x) = \ln(5x-9)^3 + \ln(2x-3)^7 = 3\ln(5x-9) + 7\ln(2x-3)$$

(iii) Let $f(x) = \ln\left((5x-9)^3(2x-3)^7\right)$. Then $f'(x) =$

$$\boxed{f'(x) = \frac{3(5)}{5x-9} + \frac{7(2)}{2x-3}}$$

QUESTION 7. (10 points)

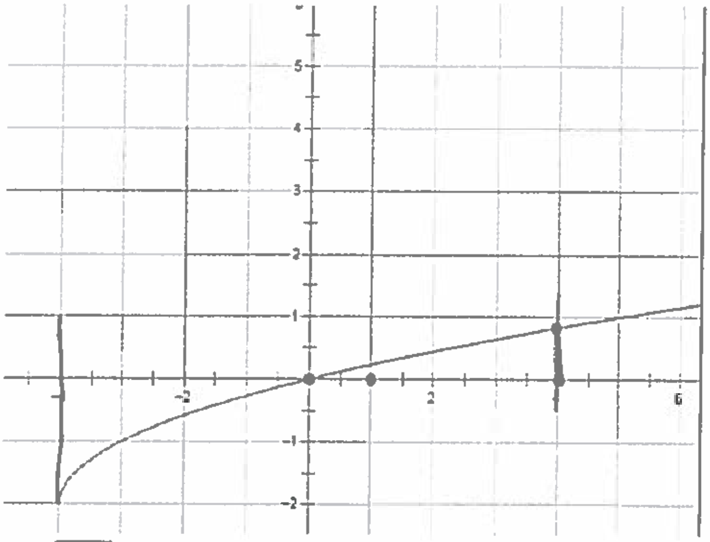
$$\int \frac{x+1}{x^2+2x+1}\,dx = \int (x+1)(x^2+2x+1)^{-1}\,dx = \boxed{\frac{1}{2}\ln|(x^2+2x+1)| + C}$$

(ii) $$\int \frac{e^x+3}{(e^x+3x+1)^2}\,dx = \int (e^x+3)(e^x+3x+1)^{-2}\,dx = \boxed{\frac{(e^x+3x+1)^{-1}}{-1} + C}$$

(iii) $$\int x^5(x+1)^2\,dx = \int x^5(x^2+2x+1)\,dx = \int x^7+2x^6+x^5\,dx =$$

$$\int x^7\,dx + 2\int x^6\,dx + \int x^5\,dx = \boxed{\frac{x^8}{8} + \frac{2x^7}{7} + \frac{x^6}{6} + C}$$

(iv) $$\int 10(2x+7)^{11}\,dx = 5\int 2(2x+7)^{11}\,dx \Rightarrow \boxed{\frac{5(2x+7)^{12}}{12} + C}$$



$$y = \sqrt{x+4} - 2$$
$$2 = \sqrt{x+4}$$
$$4 = x+4$$
$$x = 0$$

QUESTION 8.

Stare at $f(x) = \sqrt{x+4} - 2$ where $-4 \le x \le 4$. Then

a) **(6 points)** Find the area of the region bounded by the curve of $f(x)$, x-axis, and $-4 \le x \le 4$.

$$-\int_{-4}^{0} \sqrt{x+4} - 2\,dx + \int_{0}^{4} \sqrt{x+4} - 2\,dx = -\left(\frac{2}{3}(x+4)^{3/2} - 2x \Big|_{-4}^{0}\right) +$$

$$\left(\frac{2}{3}(x+4)^{3/2} - 2x \Big|_{0}^{4}\right) = -\left(\frac{16}{3} - 8\right) + \left[\left(\frac{2}{3}(8)^{3/2} - 8\right) - \frac{16}{3}\right]$$

$$\boxed{\text{Area} \approx 4.42 \text{ unit}^2}$$

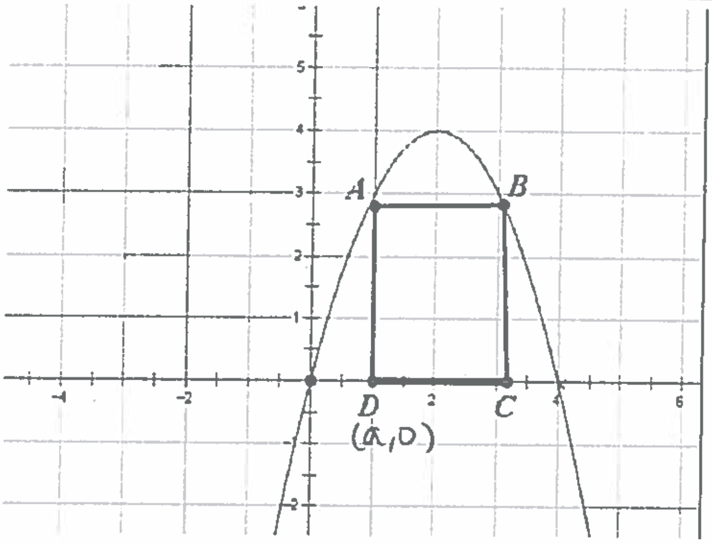
b) **(4 points)** Imagine that the region between x = 0 and x = 4 is rotated about x-axis 360 degrees. What is the volume of the object?

$$\pi \int_{0}^{4} (\sqrt{x+4} - 2)^2 dx \rightarrow \pi \int_{0}^{4} (x+4) - 4\sqrt{x+4} + 4\,dx$$

$$\Rightarrow \pi \left[\int_{0}^{4} x+8\,dx - 4\int_{0}^{4} \sqrt{x+4}\,dx\right] \Rightarrow \pi \left[\left(\frac{x^2}{2} + 8x \Big|_{0}^{4}\right) - 4\left(\frac{2(x+4)^{3/2}}{3} \Big|_{0}^{4}\right)\right]$$

$$\Rightarrow \pi \left[(40 - 0) - 4\left(\frac{2(8)^{3/2}}{3} - \frac{2(4)^{3/2}}{3}\right)\right]$$

$$\boxed{\text{Volume} \approx 0.99\pi \text{ units}^3}$$



$y = -x^2 + 4x$

$D(a,0) \to A = (a, -a^2+4a)$

$L = -a^2 + 4a$

$|OD| = |CF| = a$

$|CD| = W = 4 - 2a$

QUESTION 9. (8 points)

We want to construct a rectangle ABCD (see picture) of maximum area between the x-axis and the curve $y = -x^2 + 4x$. Find the length and the width of such rectangle. (Hint: Note that the curve intersects x-axis at $x = 0$ and at $x = 4$. Let O be the origin (0, 0) and F be (4, 0). Then |OD| = |CF|)

$A = W \times L$

$L = -a^2 + 4a$, $W = 4 - 2a$ $\to$ $A = (4-2a)(-a^2+4a)$

$A = -4a^2 + 16a + 2a^3 - 8a^2 = 2a^3 - 12a^2 + 16a$

$A' = 6a^2 - 24a + 16 = 0 \to 2(3a^2 - 12a + 8) = 0$

$\to 3a^2 - 12a + 8 = 0 \to a = \dfrac{12 \pm \sqrt{48}}{2(3)}$

$A'' = 12a - 24 \to a = \dfrac{12+\sqrt{48}}{6} \to A'' > 0$

$\to a = \dfrac{12-\sqrt{48}}{6} \to A'' < 0 \to$ Area max

when $a = \dfrac{12-\sqrt{48}}{6}$

$\boxed{L = \dfrac{8}{3}}$, $\boxed{W = \dfrac{4\sqrt{3}}{3}}$ $\to$ $\boxed{A = \dfrac{32\sqrt{3}}{9} \text{ units}^2}$

Faculty information

Ayman Badawi, Department of Mathematics & Statistics, American University of Sharjah, P.O. Box 26666, Sharjah, United Arab Emirates.
E-mail: abadawi@aus.edu, www.ayman-badawi.com